

# Mean-field solution of the parity-conserving kinetic phase transition in one dimension

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**Abstract.** A two-offspring branching annihilating random walk model, with finite reaction rates, is studied in one-dimension. The model exhibits a transition from an active to an absorbing phase, expected to belong to the  $DP2$  universality class embracing systems that possess two symmetric absorbing states, which in one-dimensional systems, is in many cases equivalent to parity conservation. The phase transition is studied analytically through a mean-field like modification of the so-called *parity interval method*. The original method of parity intervals allows for an exact analysis of the diffusion-controlled limit of infinite reaction rate, where there is no active phase and hence no phase transition. For finite rates, we obtain a surprisingly good description of the transition which compares favorably with the outcome of Monte Carlo simulations. This provides one of the first analytical attempts to deal with the broadly studied  $DP2$  universality class.

**PACS.** 02.50.Ey Stochastic processes – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 05.70.Ln Nonequilibrium and irreversible thermodynamics – 82.40.-g Chemical kinetics and reactions: special regimes and techniques

## 1 Introduction

Phase transitions occurring away from thermodynamical equilibrium constitute one of the most challenging topics in statistical physics. They appear in a host of physical systems as well as in many models in biology, chemistry, sociology, etc. Given the lack of a general theory of non-equilibrium systems, we are still living in a *taxonomic era* in this field: it would be highly desirable to reach a complete classification of the known phase transitions into universality classes, as a preliminary step to their full categorization, and identification of relevant features.

After more than twenty years of study, it has become clear that the degree of universality is much narrower away from equilibrium than it is in equilibrium transitions. Certainly, general properties such as symmetries, conservation laws, dimensionalities, etc., play a key role, as they do in equilibrium. But some other ingredients, such as microscopic dynamical details, hard core interactions, and the type of updating may, in some cases, influence the emerging critical behavior of non-equilibrium systems, making the task of theoreticians both stimulating and difficult.

One of the most robust and best studied non-equilibrium classes of transitions is *directed percolation* (DP). It includes an amazing variety of models and systems exhibiting a transition from an active to an ab-

sorbing phase [1–6], and is well characterized at a field-theoretical level by Reggeon field theory (RFT) [7,8].

DP is so robust, that identifying and classifying the nature of perturbations able to drive absorbing phase transitions away from this class has become a challenging task. Probably the best known instance of this is the  $DP2$  class, where the presence of two perfectly symmetric absorbing states ( $Z_2$  symmetry) is the main feature responsible for non-DP scaling [9]. This class includes a cellular automaton introduced by Grassberger et al. [10], interacting monomer-dimer models [11], non-equilibrium Ising models [12,13], monomer-monomer surface reaction models [14],  $Z_2$ -symmetric generalizations of DP [15] and of the contact process [16], branching annihilating random walks with conserved parity [17–19], and it has also been related to generalized versions of the Voter model [20,21]. Extensive numerical simulations led to conjecture rational exponent values for the  $DP2$  class [22], though this was later disproved by more exhaustive simulations [4].

Although the existence and robustness of the  $DP2$  class are well established from a numerical view-point, a solid theoretical understanding is still missing. Bold attempts to write down and renormalize a field theory suitable for the  $DP2$  class have been performed, but the results are not as satisfactory as they are for RFT [23,24]. In particular, the renormalization is based on a clever, but somehow uncontrolled expansion around two-different critical

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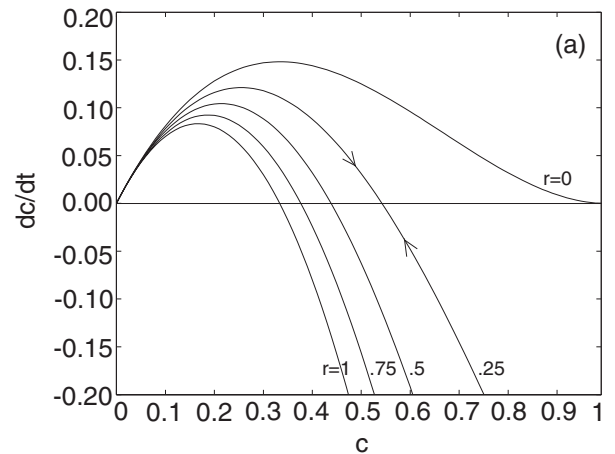
dimensions. As we shall illustrate below, straightforward mean-field approaches (cluster approximations) fail to reproduce a phase transition if too small clusters are considered, and one has to resort to large-clusters, which make the calculation complicated and not very accurate, i.e. the convergence of the series is very slow (although results can be improved if combined with *coherent-anomaly methods* [13]).

It is the purpose of this paper to shed some light on these issues, by examining models of Branching Annihilating Walks with two offspring (2-BAW) in one dimension: the 2-BAW with *finite* reaction rate exhibits a phase transition in the DP2 class [25]. Our approach is based on a mean-field modification of the *method of parity intervals*, originally introduced for the study of annihilation reactions,  $A + A \rightarrow 0$  [26–29]. A similar approach has been previously employed in conjunction with the *method of empty intervals* [26,30,31] for the analysis of other intractable models that do not conserve parity [32]. See [27,28] for a more detailed introduction to this method and its applications to different models.

The paper is organized as follows. The 2-BAW model is described in Section 2, where we also perform simple (up to two sites) cluster approximations (which fail to capture the transition). In Section 3 we present the exact solution of the 2-BAW model in the limit of *infinite* reaction rate, using the method of parity-intervals. Although there is no phase transition in this limit, the analysis serves as a basis for the mean-field like approximation presented in Section 4, for the relevant case of finite reaction rates. There we derive the approximate steady-state solution for generic parameter values, and compare it with the outcome of Monte Carlo computer simulations. We conclude with a critical discussion of our results and further developments, in Section 5.

## 2 The model and cluster approximations

The model is defined as follows. Each site of a one-dimensional lattice is either empty or singly occupied. The lattice is updated asynchronously: a randomly chosen particle attempts diffusion at rate  $\Gamma$  (probability  $\Gamma/(\Gamma + \Omega)$ ), and branching at rate  $\Omega$  (probability  $\Omega/(\Gamma + \Omega)$ ); time is increased by  $1/N$ , where  $N$  is the number of occupied (active) sites. In a diffusion attempt the particle is moved to one of its two nearest neighbors (target sites), with equal probabilities. If the target site is occupied, annihilation results with probability  $r$ : the move is effected and both particles, the diffuser and the target, are removed from the lattice. The move is rejected with probability  $1 - r$ , and the lattice state remains unchanged. In a branching attempt the particle gives birth to two new particles at the nearest neighbor sites (target sites). If either, or both of the target sites is occupied, annihilation takes place with probability  $r$ : branching is effected, and the occupied target site(s) become empty. Once again, the move is rejected with probability  $1 - r$ , and the lattice state remains unchanged.



**Fig. 1.** One-site cluster approximation. Flow diagram, according to equation (1), with  $dc/dt = 0$ . This shows that the root  $c_s = 0$  is unstable, while the root  $c_s > 0$  is stable. Results from the two-site approximation are very similar. The stationary solution as a function of  $r$  is plotted in Figure 2.

In the diffusion-controlled limit of infinite reaction rate ( $r = 1$ ), the 2-BAW model is known to possess only a steady absorbing phase [33]. However, for finite rates ( $r < 1$ ), a transition belonging to the DP2 class, from an absorbing state, at  $r < r_c$ , to an active phase, at  $r > r_c$ , is found in numerical simulations [25]. We now demonstrate that simple mean-field approximations fail to capture this transition.

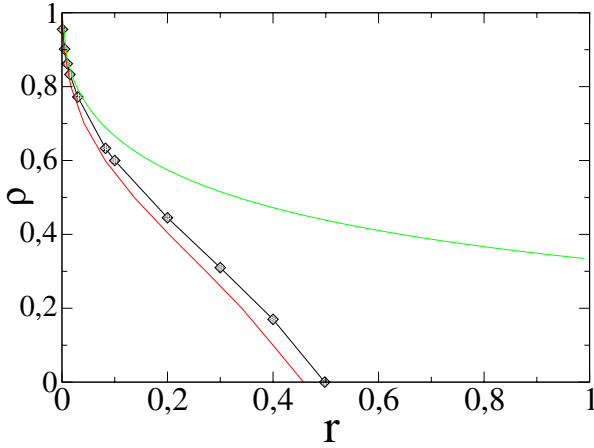
The simplest conceivable mean-field theory is that of the one-site approximation, obtained by neglecting all correlations between the states of different sites. For example, if the probability of one site being occupied is equal to the concentration  $c$ , then the probability of finding one occupied site followed immediately by an empty site is  $c(1 - c)$ . There are only three events that lead to a change in particle concentration: (i) annihilation of two particles by diffusion,  $\bullet\bullet \rightarrow \circ\circ$ ; (ii) creation of two particles via birth,  $\circ\circ \rightarrow \bullet\bullet$ ; and (iii) annihilation of two particles due to birth onto previously occupied sites,  $\bullet\bullet\bullet \rightarrow \circ\bullet\circ$ . Note that the birth process  $\circ\bullet\bullet \rightarrow \bullet\bullet\circ$  (and its mirror-symmetric image) does not alter the concentration. However, processes (i) and (iii) occur with restricted probability  $r$ . Thus

$$\frac{d}{dt}c = -2rc^2 + 2c(1 - c)^2 - 2rc^3, \quad (1)$$

where we took  $\Gamma = \Omega = 1$ . The first, second and third terms on the r.h.s. correspond to (i), (ii) and (iii), respectively.

For the steady state, we set  $dc/dt = 0$  and choose the stable root (Fig. 1),  $c_s = (2 + r - \sqrt{8r + r^2})/(2 - 2r)$ . We conclude that according to the one-site approximation the system always evolves to an active phase, with  $1/3 < c_s < 1$  (Fig. 2).

The two-site approximation provides the next level of complexity. Let  $p$  be the conditional probability for a site to be occupied, given that the adjacent site (to its left) is



**Fig. 2.** Order parameter  $\rho$  as a function of  $r$  as obtained from (a) Monte Carlo simulations (diamonds), (b) parity interval approximation (lowermost curve), and (c) one-site cluster approximation (uppermost curve): the curve shows the stable root (see Fig. 1), (observe that  $\rho$  corresponds to  $c_s$  in the cluster approximation section). Note the absence of a transition point in the third case.

occupied. Let  $q$  be the conditional probability for a site to be occupied, given that the site to its left is empty. Thus, the probability of the events  $\bullet\bullet$ ,  $\bullet\circ$ ,  $\circ\bullet$ ,  $\circ\circ$  is  $cp$ ,  $c(1-p)$ ,  $(1-c)q$ ,  $(1-c)(1-q)$ , respectively. Note that since  $\bullet\circ$  and  $\circ\bullet$  are equally likely, we have,

$$c(1-p) = (1-c)q. \quad (2)$$

(The same relation may be derived from the fact that  $\Pr(\bullet) = \Pr(\bullet\bullet) + \Pr(\circ\bullet)$ .) Equation (1) is now rewritten as

$$\frac{d}{dt}c = -2rcp + 2(1-c)q(1-p) - 2rcp^2, \quad (3)$$

where the terms on the r.h.s. correspond to the same events as before. Setting  $dc/dt = 0$  and using the relation (2), we obtain, for the steady state,

$$0 = -2rcp + 2c(1-p)^2 - 2rcp^2.$$

We conclude that either  $c_s = 0$  or  $p_s = (2 + r - \sqrt{8r + r^2}) / (2 - 2r)$ . In the latter case, an additional evolution equation, for the event  $\bullet\bullet$ , provides the missing value of  $c_s$ :  $c_s > 0$  for all  $r$ . The result agrees closely with that of the one-site approximation, with deviations not larger than 20%. Thus, also the two-site approximation fails to predict the transition. Considering larger clusters, a phase transition can be generated [13, 34], but a large set of coupled equations (which enlarges with cluster-size) has to be solved, the accuracy is not good, and the results are expected to be valid only up to the cluster-size. In the next section we present an alternative method intended to overcome these difficulties.

### 3 Parity intervals and exact solution for $r = 1$

We now turn to a different approach, that of the method of parity intervals [26–29]. We first present the exactly

soluble case of  $r = 1$ , for which there is no transition. The exact approach followed here serves as a basis for a mean-field like approximation, for the more interesting case of  $r < 1$  — an approximation that does capture the transition and reproduces kinetic details surprisingly well.

Let  $G_n(t)$  be the probability that (in an homogeneous system) an arbitrary segment of  $n$  consecutive sites contains an even number of particles at time  $t$ . A site can be either empty or occupied by a single particle, so the probability that a site is occupied, i.e., the particle density, is

$$\rho(t) = 1 - G_1(t). \quad (4)$$

Since the dynamic rules of the two-offspring BAW conserve parity, the only way that  $G_n$  might change is when: (i) particles at the edge of the segment hop outside or branch, or (ii) particles just outside of the segment hop inside or branch. Let  $\overline{\bullet\bullet\bullet}$  and  $\overline{\bullet\bullet\bullet}$  represent segments of even and odd number of particles, respectively. Then, the changes of  $G_n$  can be described schematically by the events:

$$\begin{aligned} \frac{d}{dt}G_n = & \overline{\bullet\bullet\bullet} \xleftarrow{n} + \overline{\bullet\bullet\bullet} \xrightarrow{n-1} - \overline{\bullet\bullet\bullet} \xleftarrow{n} - \overline{\bullet\bullet\bullet} \xrightarrow{n-1} \\ & + \overline{\bullet\bullet\bullet} \xleftarrow{n} + \overline{\bullet\bullet\bullet} \xrightarrow{n-1} - \overline{\bullet\bullet\bullet} \xleftarrow{n} - \overline{\bullet\bullet\bullet} \xrightarrow{n-1}. \end{aligned} \quad (5)$$

Arrows in the first four terms indicate the hopping of a particle to the left or right. In the last four terms, the arrows indicate branching. For example, the first term in (5) represents the event that a particle outside of an  $n$ -segment of odd parity jumps in, thus creating an  $n$ -segment of even parity.

For the case of immediate reactions,  $r = 1$ , the processes indicated in (5) occur regardless of the state of target sites. It is easy to show (see [26–29]) that:

$$F_n \equiv \Pr(\overline{\bullet\bullet\bullet}) = \frac{1}{2} [(1 - G_1) + (G_n - G_{n+1})], \quad (6a)$$

$$H_n \equiv \Pr(\overline{\bullet\bullet\bullet}) = \frac{1}{2} [(1 - G_1) - (G_n - G_{n+1})]. \quad (6b)$$

Using these relations, equation (5) becomes

$$\frac{d}{dt}G_n(t) = 2(\Gamma + \Omega)(G_{n-1} - 2G_n + G_{n+1}). \quad (7)$$

$\Gamma$  and  $\Omega$  are the rates of hopping and branching, respectively, and the factor of 2 accounts for the events in (5) taking place also at the left edge of the interval. The case of  $n = 1$  requires a special equation, since  $G_0$  is undefined. Taking into account all the ways  $G_1$  might change, we find

$$\frac{\partial}{\partial t}G_1(t) = 2\Gamma(1 - 2G_1 + G_2) + 2\Omega(G_2 - G_1). \quad (8)$$

Thus, equation (7) may be understood to be valid also for  $n = 1$ , provided that one uses the boundary condition

$$G_0 = \frac{\Gamma + \Omega G_1}{\Gamma + \Omega}. \quad (9)$$

$$\begin{aligned}
 \overline{\overline{\bullet}}^n \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} &= \overline{\overline{\bullet}}^{n-1} \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} \circ \bullet + r \overline{\overline{\bullet}}^{n-1} \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} \bullet \bullet + r \overline{\overline{\bullet}}^{n-1} \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} \bullet \bullet \bullet + r \overline{\overline{\bullet}}^{n-1} \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} \bullet \bullet \bullet \bullet \\
 &= (1-r) \overline{\overline{\bullet}}^{n-1} \circ \bullet \bullet + r \overline{\overline{\bullet}}^{n-1} \bullet \bullet \bullet + r \overline{\overline{\bullet}}^{n-1} \bullet \bullet \bullet \bullet + r \overline{\overline{\bullet}}^{n-1} \bullet \bullet \bullet \bullet \bullet \\
 &= s \overline{\overline{\bullet}}^{n-1} \circ \bullet \bullet + (1-s) \overline{\overline{\bullet}}^n \bullet.
 \end{aligned} \tag{12a}$$

Additionally, since the  $G_n$  are probabilities, we have

$$0 \leq G_n(t) \leq 1. \tag{10}$$

Equation (7), with the boundary conditions (9), (10) may be analyzed exactly for a variety of initial conditions [27]. Here we merely observe that the only steady state solution supported by these equations is  $G_n = 1$ ,  $n = 1, 2, \dots$ , corresponding to the absorbing state  $\rho_s = 0$ . Indeed, the 2-BAW model with  $r = 1$  lacks an active phase and exhibits no transition.

### 4 Parity intervals approximation for the general case

Consider now the case of finite reaction probability,  $r < 1$ . A transition about some critical probability value  $r_c$  is known to take place, from the absorbing state into an active (non-empty) phase [25], and our goal is to capture this transition, if only in an approximate fashion. We shall assume that  $\Gamma = \Omega = 1$ , without loss of generality [35].

Consider the first process on the r.h.s. of (5). For  $r < 1$ , there is a difference in the reaction rate depending on whether the target site is empty or occupied:

$$\overline{\overline{\bullet}}^n \leftarrow = \overline{\overline{\bullet}}^{n-1} \leftarrow \circ \bullet + r \overline{\overline{\bullet}}^{n-1} \leftarrow \bullet \bullet.$$

Unfortunately, events such as  $\overline{\overline{\bullet}} \circ \bullet$  and  $\overline{\overline{\bullet}} \bullet \bullet$  cannot be expressed in closed form in terms of the  $G_n$ . We therefore rewrite the process in a way that the terms associated with  $r < 1$ , and which cannot be expressed in closed form, appear as a perturbation, proportional to  $s \equiv 1 - r$ :

$$\begin{aligned}
 \overline{\overline{\bullet}}^n \leftarrow &= \overline{\overline{\bullet}}^{n-1} \leftarrow \circ \bullet + r \overline{\overline{\bullet}}^{n-1} \leftarrow \bullet \bullet \\
 &= \overline{\overline{\bullet}}^{n-1} \circ \bullet + \overline{\overline{\bullet}}^{n-1} \bullet \bullet - (1-r) \overline{\overline{\bullet}}^{n-1} \bullet \bullet \\
 &= \overline{\overline{\bullet}}^n \bullet - s \overline{\overline{\bullet}}^{n-1} \bullet \bullet.
 \end{aligned} \tag{11a}$$

Likewise, the remainder of the diffusion events in (5) may be rewritten as

$$\overline{\overline{\bullet}}^{n-1} \rightarrow = \overline{\overline{\bullet}}^{n-1} \bullet - s \overline{\overline{\bullet}}^{n-1} \bullet \bullet, \tag{11b}$$

$$\overline{\overline{\bullet}}^n \leftarrow = \overline{\overline{\bullet}}^n \bullet - s \overline{\overline{\bullet}}^{n-1} \bullet \bullet, \tag{11c}$$

$$\overline{\overline{\bullet}}^{n-1} \rightarrow = \overline{\overline{\bullet}}^{n-1} \bullet - s \overline{\overline{\bullet}}^{n-1} \bullet \bullet. \tag{11d}$$

The branching events too require close inspection of the target sites. For example, we rewrite the first branching event of (5) in a perturbative fashion:

See equation (12a) above.

The remainder of the branching terms are similarly expressed as

$$\overline{\overline{\bullet}}^{n-1} \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} = s \overline{\overline{\bullet}}^{n-2} \circ \bullet \bullet + (1-s) \overline{\overline{\bullet}}^{n-1} \bullet, \tag{12b}$$

$$\overline{\overline{\bullet}}^n \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} = s \overline{\overline{\bullet}}^{n-1} \circ \bullet \bullet + (1-s) \overline{\overline{\bullet}}^n \bullet, \tag{12c}$$

$$\overline{\overline{\bullet}}^{n-1} \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \end{array} \bullet = s \overline{\overline{\bullet}}^{n-2} \circ \bullet \bullet + (1-s) \overline{\overline{\bullet}}^{n-1} \bullet. \tag{12d}$$

So far everything is exact. In order to proceed, we approximate the terms proportional to  $s$  in the simplest possible way, by neglecting correlations. Thus, for example, we write

$$\Pr(\overline{\overline{\bullet}}^{n-1} \bullet \bullet) \approx \Pr(\overline{\overline{\bullet}}^{n-1} \bullet) \Pr(\bullet) = F_{n-1}(1 - G_1), \tag{13}$$

for the problematic term in (11a), and similar expressions for the ones in (11b)–(11d). For the problematic terms proportional to  $s$  in (12), we introduce the notation

$$\Pr(\overline{\overline{\bullet}}^n \circ) = f_n, \quad \Pr(\overline{\overline{\bullet}}^{n-1} \circ) = h_n,$$

and

$$\Pr(\bullet \bullet) = x, \quad \Pr(\bullet \circ) = \Pr(\circ \bullet) = y, \quad \Pr(\circ \circ) = z,$$

and approximate in the same spirit as above,

$$\Pr(\overline{\overline{\bullet}}^{n-1} \circ \bullet \bullet) \approx \Pr(\overline{\overline{\bullet}}^{n-1} \circ) \Pr(\bullet \bullet) = h_{n-1}y, \tag{14}$$

and similarly for the other terms. Since  $y + z = G_1$ ,  $x + z = G_2$ , and  $x + 2y + z = 1$ , it follows that  $x = \frac{1}{2}(1 - 2G_1 + G_2)$ ,  $y = \frac{1}{2}(1 - G_2)$ , and  $z = \frac{1}{2}(-1 + 2G_1 + G_2)$  are expressible in terms of the  $G_n$  in closed form. Since  $f_n + F_n = G_n$  and  $h_n + H_n = 1 - G_1$ , it follows that  $f_n = \frac{1}{2}(-1 + G_1 + G_n + G_{n+1})$  and  $h_n = \frac{1}{2}(1 + G_1 - G_n - G_{n+1})$  are too given in terms of the  $G_n$  in closed form.

We are now ready to write down a closed evolution equation for  $G_n$ . Starting from equation (5), using the representations of equations (11) and (12) with their respective approximations, collecting terms and rearranging, we obtain

$$\begin{aligned}
 \frac{d}{dt} G_n &= \frac{1}{2} s (1 - G_2) G_{n-2} + (2 - 3s + 2sG_1) G_{n-1} \\
 &\quad + \frac{1}{2} (-8 + 7s - 4sG_1 + sG_2) G_n + (2 - s) G_{n+1},
 \end{aligned} \tag{15}$$

$$\lambda_0 = 1, \quad \lambda_{\pm} = \frac{4 - 5s + 4sG_1 - sG_2 \pm \sqrt{(4 - 5s + 4sG_1 - sG_2)^2 + 8s(2 - s)(1 - G_2)}}{8 - 4s}. \quad (18)$$

valid for  $n \geq 3$ . As expected, this reduces to (7), with  $\Gamma = \Omega = 1$ , in the limit  $s \rightarrow 0$ . The equation for  $n = 2$  is exactly the same as (15), provided that one adopts the boundary condition

$$G_0 = 1. \quad (16)$$

$G_1$  requires a separate evolution equation that we find by considering all the events that contribute to  $dG_1/dt$ , and write down, in the spirit of equation (15), as

$$\begin{aligned} \frac{d}{dt}G_1 &= 2(\overset{\leftarrow}{\bullet\bullet} + \overset{\rightarrow}{\bullet\bullet} - \overset{\leftarrow}{\circ\circ} + \overset{\rightarrow}{\circ\circ} - \overset{\nwarrow}{\bullet\bullet} - \overset{\swarrow}{\bullet\bullet} + \overset{\nwarrow}{\circ\circ} + \overset{\swarrow}{\circ\circ}) \\ &\approx 2\{(1-s)x + [(1-G_1) - sx] \\ &\quad - y + (1-s)x - [(1-s)y + syG_1]\} \\ &= 2 - 6G_1 + 4G_2 \\ &\quad + s(-2 + 5G_1 - 4G_2 + G_1G_2). \end{aligned} \quad (17)$$

To compute the steady state, we set  $dG_n/dt = 0$ . Equation (15) yields a recursion relation for the  $G_n$ , with constant coefficients (that depend partly on  $G_1$  and  $G_2$ ). Applying the ansatz  $G_n = \lambda^n$  we find a cubic equation for  $\lambda$ , with roots

See equation (18) above.

Thus  $G_n$  has the general solution

$$G_n = A\lambda_+^n + B\lambda_-^n + C,$$

where  $A, B, C$  are constants to be determined from boundary conditions. Since  $0 < s, G_1, G_2 < 1$  (in the active phase) it follows that  $|\lambda_{\pm}| < 1$ , and  $C = \lim_{n \rightarrow \infty} G_n \equiv G_{\infty}$ . Suppose that the initial distribution of particles is random, at density  $\rho_0$ , then  $G_n(t=0) = \frac{1}{2} + \frac{1}{2}(1-2\rho_0)^n$ , and  $G_{\infty}(0) = 1/2$  [27]. Because our model conserves parity, it follows that  $C = 1/2$ . Furthermore, the boundary condition (16) implies  $B = \frac{1}{2} - A$ . The remaining coefficient,  $A$ , could be found from the relations

$$G_1 = A\lambda_+ + \left(\frac{1}{2} - A\right)\lambda_- + 1/2, \quad (19a)$$

$$G_2 = A\lambda_+^2 + \left(\frac{1}{2} - A\right)\lambda_-^2 + 1/2, \quad (19b)$$

$$G_2 = \frac{2s - 2 + (6 - 5s)G_1}{4 - 4s + sG_1}. \quad (19c)$$

The first two relations are required for self-consistency, since  $\lambda_{\pm}$  depend on  $G_1$  and  $G_2$  (as well as on  $s$ ). Equation (19c) is derived from (17), with  $dG_1/dt = 0$ . Because equations (19) are unwieldy, in practice we set a numerical value for  $G_1$  and search for  $(A, s, G_2)$  that satisfy (19), using Mathematica. We thus find a kinetic phase transition which we next compare to simulations.

The critical point  $r_c = 1 - s_c$ , and the behavior of the order parameter in its vicinity, can be obtained analytically, by perturbing (19) about the values  $A = 0$ ,

$G_1 = 1$ . We thus find  $r_c = \frac{1}{6}\sqrt{33} - \frac{1}{2} = 0.457427\dots$  and  $\rho_s \sim (r_c - r)^{\beta}$ , with  $\beta = 1$ . The value of  $\beta = 1$  is in accordance with the expected from a mean-field like solution.

We can also obtain analytically the behavior of the order parameter in the neighborhood of  $r = 0$ . The answer,  $c_s \sim 1 - \sqrt{2r}$ , is the same as the one obtained from the one-site and two-site cluster approximations, and seems to fit the numerical data from simulations exactly. Clearly, the role of fluctuations diminishes as  $r$  decreases away from the critical point, and it is conceivable that their impact is negligible in the limit of zero reaction.

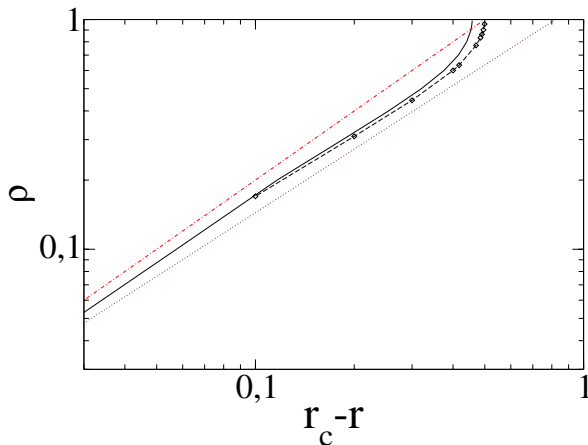
### Comparison to simulations

In order to test the limit of validity of the above calculations, we have performed a Monte Carlo simulation of the 2-BAW model. We use a system size  $L = 10^4$ , and implement the previously discussed rules. The initial configuration consists of a randomly half-occupied lattice; we have verified that the obtained long-time asymptotic results are insensitive to variations of the initial condition. Plotting the average density as a function of time in a double logarithmic plot, we identify the critical point with the value of  $r$  for which a separatrix, between curves that tend asymptotically to a constant (active phase) and curves that bend downwards (absorbing phase) converging asymptotically to a  $t^{-1/2}$  decay [36] is obtained. We thus find  $r_c = 0.495(10)$ . From the slope of the plot the density decay critical exponent  $\theta = 0.28(1)$  (extending for about four decades), is determined. It is in excellent agreement with previous measurements in the DP2 class [4]. The stationary density values are represented in Figure 2.

A point worth emphasizing is that finite-size corrections differ from those usually encountered in systems with absorbing states: Rather than finding a larger stationary order parameter for smaller systems, as is common, here one observes a faster decay to the absorbing phase. Owing to this the exponent ratio  $\beta/\nu$  cannot be determined from finite-size scaling analysis, unless measurements are restricted to surviving runs as done in [37].

We have also measured the order-parameter critical exponent, by plotting the average stationary density as a function of the distance to the critical point (Fig. 3). Our finding of  $\beta = 0.92(5)$  is again in good agreement with the commonly accepted value [4]. Other exponents can certainly be measured, but with  $\theta$  and  $\beta$  we can already guarantee that the model is indeed in the DP2 class.

A more stringent test of the mean-field parity interval approximation is obtained by comparing its predictions for  $G_n$  to the numerical results of Monte Carlo simulations, for different values of the control parameter  $r$  (Fig. 4). In all cases, we expect  $G_n \rightarrow 1/2$  for large  $n$ , since  $G_{\infty} = 1/2$  and parity is conserved. The relevant question is how well the approximation captures the  $G_n$



**Fig. 3.** Log-log plot of the order parameter vs. the distance to the critical point. Shown are simulation results (curve with diamonds) compared to the prediction from the parity interval approximation (solid curve). Slopes of 1 (top, dashed line) and 0.92 (bottom, dotted line) are shown for comparison. The latter represents the accepted value of the order parameter exponent  $\beta$  for the DP2 universality class in  $d = 1$ . Nearby criticality, the mean-field solution converges to  $\beta = 1$ . Surprisingly, the mean-field slope agrees quite well with the expected value of 0.92, when  $r_c - r \gtrsim 0.1$ .

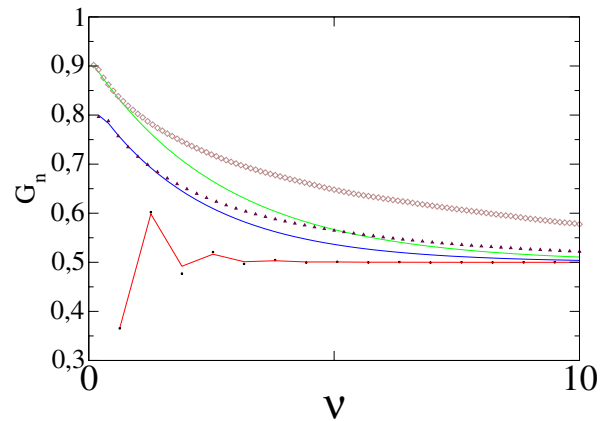
for  $n$  small. Because the typical distance between particles grows as one approaches criticality, we plot the results against  $\nu = \rho n$  rather than  $n$ , in order to compare them better.

For small values of  $r$  (as shown in Fig. 4), deep into the active phase, spatial correlations play a minimal role and the theoretical prediction matches simulations remarkably well, for all  $\nu$ . As  $r$  is raised toward the critical point, correlations play a larger role and the fit worsens. For example for  $r = 0.38$  ( $\rho = 0.2$ ) the theoretical curve fits experiments well only up to  $\nu \approx 2$  (and, of course, also at  $\nu \gg 1$ ). Closer still to the critical point, at  $r = 0.45$  ( $\rho = 0.1$ ), the theoretical prediction is good only up to  $\nu \approx 1$ . The large deviations observed for  $\nu \gtrsim 1$  in this last case indicate that the state of the system near the DP2 transition, despite being dilute, is highly ordered and correlated.

The approximate curves converge to the exact value for  $G_n \rightarrow 1/2$  for asymptotically large values of  $\nu$ .

## 5 Discussion and future perspectives

We have presented an approximation, based on the method of parity intervals, for the analysis of the DP2 transition observed in the 2-BAW model in one dimension with finite reaction rates. The key ingredient of our approach is that it respects the parity conservation implicit in the dynamical rules of the 2-BAW model: parity conservation is here responsible for the  $Z_2$  symmetry that underlies transitions in the DP2 class, and provides a surprisingly good description of the transition at a mean-field level.



**Fig. 4.**  $G_n$  in the active phase for three different stationary activity values: (from the bottom to the top) 0.633, 0.2, and 0.1, as computed in a) Monte Carlo simulation for  $r = 0.082$ ,  $r = 0.38$  and  $r = 0.45$  respectively (curves with symbols) and b) interval approximation (continuous lines). Instead of plotting as a function on  $n$  in order to make more meaningful the comparison we use  $\nu$ , defined as  $n$  divided the average interparticle distance in each case.

An interesting finding is the fact that spatial correlations play no role in the limit of zero reaction rate,  $r \rightarrow 0$ . In this “saturation” limit, the dependence of the order parameter upon the critical field,  $\rho \sim 1 - \sqrt{2r}$ , is captured even by the simplest one-site cluster approximation.

Extensive numerical simulations of the DP2 transition suggest a value of  $\beta = 0.92(2)$  for the order parameter critical exponent [4]. On the other hand, an exact value  $\beta = 1$  was conjectured by Jensen and later supported by numerical simulations combined with Padé approximants [16]. Regrettably, we have little to add to this controversy. Our mean-field results away from criticality ( $r_c - r \gtrsim 0.1$ ) are consistent with  $\beta = 0.92$ , while  $\beta \rightarrow 1$  as we approach the transition (Fig. 3). However, the large spatial correlations observed near criticality are not faithfully modeled by our mean-field like approach (Fig. 4), and the values we obtain for  $r_c - r \lesssim 0.1$  are unreliable. It would be desirable to improve our approximation so as to better model regions closer to  $r_c$ . Our attempts in this direction have been futile, so far. Arguably, the simplest improvement would be to replace the approximation of equation (13) by the more accurate:

$$\Pr(\overline{\overline{\overline{\overline{\bullet\bullet}}}}) \approx \frac{\Pr(\overline{\overline{\overline{\bullet}}}) \Pr(\bullet\bullet)}{\Pr(\bullet)},$$

and likewise elsewhere. However, on employing this scheme, instead of an improved result we find that the transition disappears. A similar phenomenon is known to occur also in mean-field cluster approximations, where sometimes increasing the cluster size yields lower quality predictions.

Other interesting open prospects include using the parity interval approximation for the analysis of dynamical aspects of the DP2 transition. We have here systematically assumed that all time derivatives are zero, thereby

accessing the steady state alone. Analyzing the very same equations with the full time dependence built in should yield a prediction for the order parameter time decay. Additionally, the equations could be modified to describe a spatially inhomogeneous system. In that case, one could study the dynamics of spreading.

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35. The same transition is observed whether one fixes  $r$  and treats  $\Gamma/\Omega$  as a critical field, or whether one fixes  $\Gamma/\Omega$  and takes  $r$  to be the critical field. Here we adopt the latter, with  $\Gamma/\Omega = 1$
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